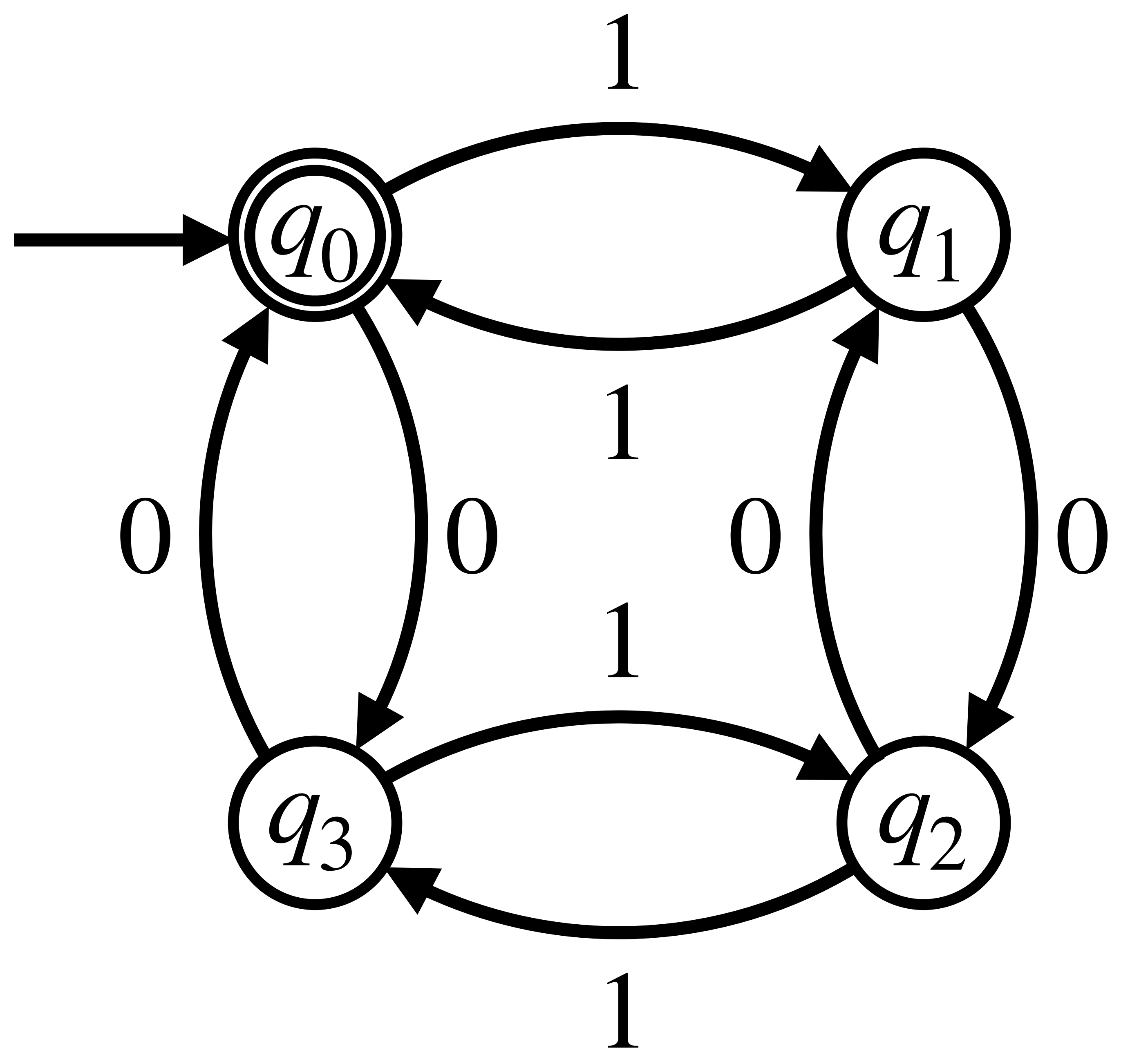
Lecture 37

More DFAs, Pumping Lemma

More DFAs

Example: Construct a DFA whose language is the set of binary strings that contain even number of 0s and even number of 1s.



- q_0 corresponds to even number of 0s and $1s$.
- q_1 corresponds to even number of 0s and odd number of $1s$.
- q_2 corresponds to odd number of 0s and $1s$.
- q_3 corresponds to odd number of 0s and even number of $1s$.

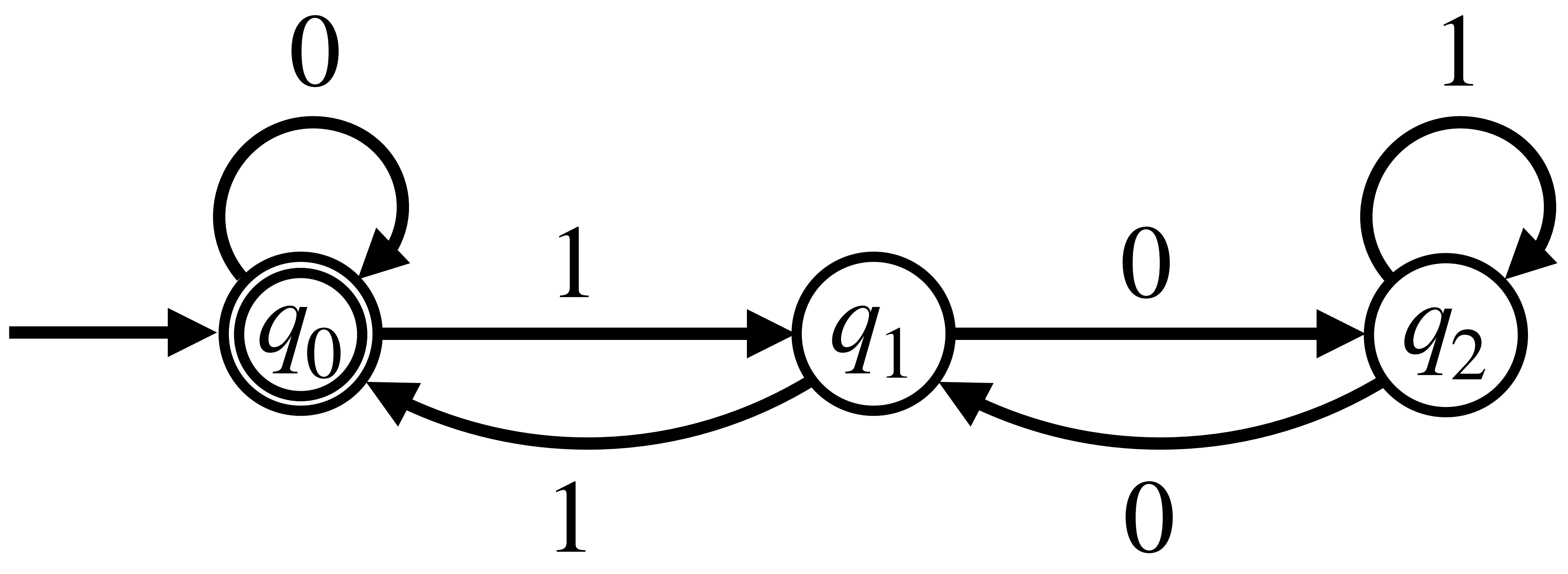
More DFAs

Example: Construct a DFA whose language is the set of binary strings whose decimal representation is divisible by 3.

Idea: We process the input string w from left to right and keep track of $dec(x)\ \%\ 3$, where $dec(x)$ is the decimal representation of the prefix x of w that we have processed.

Suppose $dec(x)\ \%\ 3 = k$, where $k \in \{0,1,2\}$. Then,

- $dec(x1)\ \%\ 3 = (2 * dec(x) + 1)\ \%\ 3 = (2 * k + 1)\ \%\ 3$
- $dec(x0)\ \%\ 3 = (2 * dec(x) + 0)\ \%\ 3 = (2 * k + 0)\ \%\ 3$



- q_0 corresponds to $dec(x)\ \%\ 3 = 0$.
- q_1 corresponds to $dec(x)\ \%\ 3 = 1$
- q_2 corresponds to $dec(x)\ \%\ 3 = 2$

More DFAs

Example: Construct a DFA that decides $L = \{0^n 1^n \mid n \geq 0\}$.

Does not seem possible as we need to keep a count of number of 0s but we can have only a fixed/finite amount of states in the DFA.

Proof of why there is no DFA for $L = \{0^n 1^n \mid n \geq 0\}$:

Suppose there is a DFA M of k states that decides L.

Consider the string $w = 0^k 1^k$ which is in L and also accepted by M.

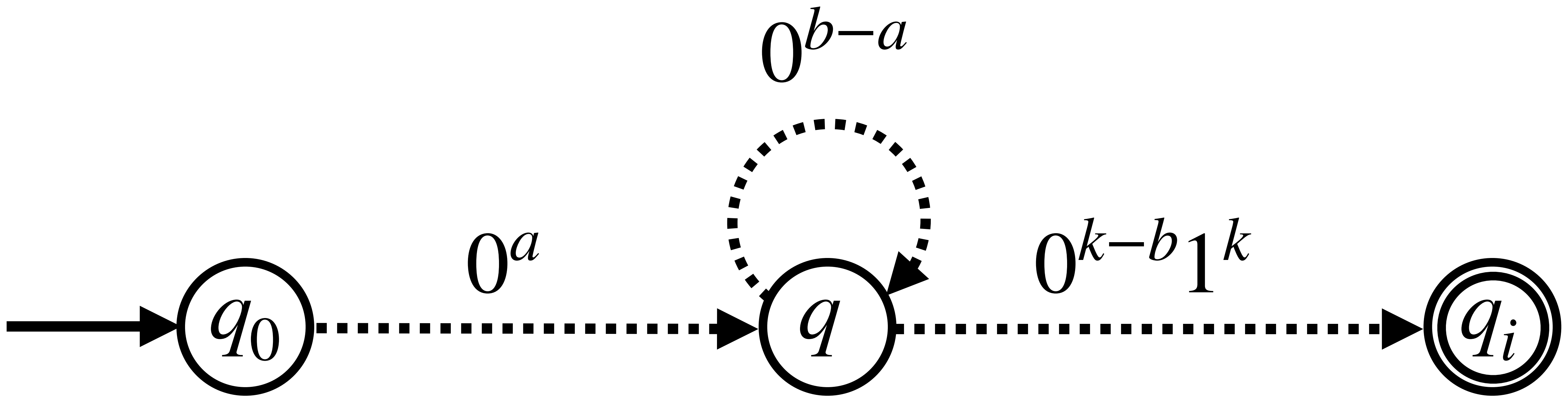
q_0	q_{i_1}	q_{i_2}	q_{i_3}	q_{i_4}	$\dots$	q_{i_k}	← *Current states after processing 0^i*
ϵ	0	0^2	0^3	0^4	$\dots$	0^k	

By pigeonhole principle, $\exists a, b$, where $0 \leq a < b$, such that after processing 0^a and 0^b, M's current state is the same, say q.

More DFAs

$$w = 0^k1^k = \overbrace{\underbrace{000\ldots0}_{0^a}\ldots0}^{0^b}\ldots01^k = 0^a0^{b-a}0^{k-b}1^k$$

We will construct a string $w' \notin L$ (from w) that gets accepted by M. Hence, a contradiction.



If M accepts $w = 0^a0^{b-a}0^{k-b}1^k$, then it will accept $w' = 0^a0^{k-b}1^k$ as well.

In general, $0^a0^{(b-a).l}0^{k-b}1^k$, for $l \geq 0$, will be accepted by M. ■

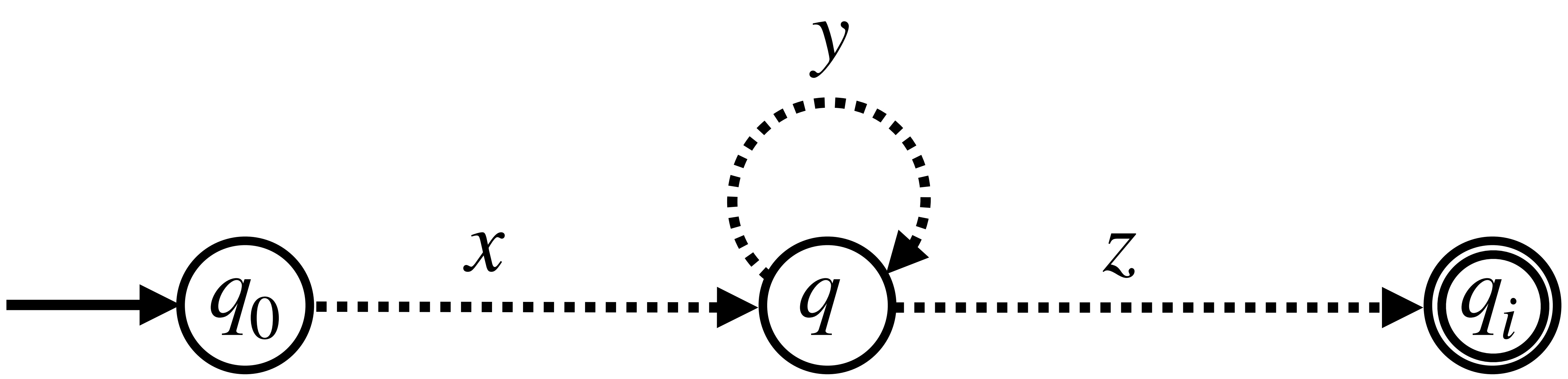
Pumping Lemma

Definition: A language L is **regular** if the there exists a DFA M that decides L.

Pumping Lemma: Let L be a regular language with a DFA of n states. Then for every string w in L such that $|w| \geq n$, we can break w into three strings, $w = xyz$, such that:

1. $y \neq \epsilon$
2. $|xy| \leq n$
3. For all $k \geq 0$, the string xy^kz is also in L.

Proof: Similar to the previous proof.



Proving Non-regularity Using Pumping Lemma

Pumping Lemma: Let L be a regular language with a DFA of n states. Then for every string w in L such that $|w| \geq n$, we can break w into three strings, $w = xyz$, such that:

1. $y \neq \epsilon$
2. $|xy| \leq n$
3. For all $k \geq 0$, the string xy^kz is also in L.

How to prove a language L non-regular?

- Assume that L is regular and has a DFA of n states.
- Pick a string $w \in L$ such that $|w| \geq n$.
- Show that it is impossible to split w into three string, $w = xyz$, such that all 3 conditions satisfy.

Proving Non-regularity Using Pumping Lemma

Example: Prove that $L = \{0^n 1^n \mid n \geq 0\}$ is non-regular.

Solution: Suppose L is regular and has a DFA of n states.

Let $w = 0^n 1^n$ be a string in L. Clearly, $|w| = 2n \geq n$.

According to Pumping Lemma, w can be split into 3 strings, $w = xyz$, such that

$1)\ y \neq \epsilon$, $2)\ |xy| \leq n$, $3)$ For all $k \geq 0$, the string $xy^k z$ is also in L.

We show now that it is impossible to split w so that it satisfies all three conditions.

- If y contains neither 1s nor 0s, split will violate condition 1.
- If y contains some 1s, split will violate condition 2.
- If y contains only 0s, then $xyyz$ will not be in L as $xyyz$ contains more 0s than 1s.

  Hence, it will violate condition 3.

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Proving Non-regularity Using Pumping Lemma

Example: Prove that $L = \{w \mid w \text{ is a palindromic binary string}\}$ is non-regular.

Solution: Suppose L is regular and has a DFA of n states.

Let $w = 0^n10^n$ be a string in L.

According to Pumping Lemma, w can be split into 3 strings, $w = xyz$, such that

$1)\ y \neq \epsilon$, $2)\ |xy| \leq n$, $3)$ For all $k \geq 0$, the string xy^kz is also in L.

We show now that it is impossible to split w so that it satisfies all three conditions.

- If y contains neither 1 nor 0s, split will violate condition 1.
- If y contains 1 or 0s after 1, split will violate condition 2.
- If y contains only 0s before 1, then the $xyyz$ will not be in L as $xyyz$ is not a palindrome. Hence, it will violate condition 3.

■

Proving Non-regularity Using Pumping Lemma

Example: Prove that $L = \{w \mid w \text{ is a binary string that contains equal number of } 0\text{s and } 1\text{s}\}$ is non-regular.

Solution: Suppose L is regular and has a DFA of n states.

Let $w = 0^n 1^n$ be a string in L. Clearly, $|w| = 2n \geq n$.

Repeat the argument for $L = \{0^n 1^n \mid n \geq 0\}$. ■